Chapter 7: DIMENSIONAL ANALYSIS

INTRODUCTION

Dimensional analysis is a mathematical technique which makes use of the study of the dimensions for solving several engineering problems. Each physical phenomenon can be expressed by an equation giving relationship between different quantities, such quantities are dimensional and non-dimensional. Dimensional analysis helps in determining a systematic arrangement of the variables in the physical relationship, combining variables to parameters. It is based on the non-dimensional principle of dimensional and uses the dimensions variables affecting the phenomenon. Dimensional analysis has become an important tool for analysing fluid flow problems. It is specially useful in presenting experimental results in a concise form.

Uses of dimensional analysis:

The uses of dimensional analysis may be summarized as follows:

1. To test the dimensional homogeneity of any equation of fluid motion.
2. To derive rational formulae for a flow phenomenon.
3. To derive equations expressed in terms of non-dimensional parameters to show the relative significance of each parameter.
4. To plan model tests and present experimental results in a systematic manner, thus making it possible to analyze the complex fluid flow phenomenon.

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Advantages of dimensional analysis:

1. It expresses the functional relationship between the variables in dimensionless terms.
2. In hydraulic model studies it reduces the number of variables involved in a physical phenomenon, generally by three.
3. By the proper selection of variables, the dimensionless parameters can be used to make certain logical deductions about the problem.
4. Design curves, by the use of dimensional analysis, can be developed from experimental data or direct solution of the problem.
5. It enables getting up a theoretical equation in a simplified dimensional form.
6. Dimensional analysis provides partial solutions to the problems that are too complex to be dealt with mathematically.
7. The conversion of units of quantities from one system to another is facilitated.

7.2. DIMENSIONS

The various physical quantities used in fluid phenomenon can be expressed in terms of fundamental quantities or primary quantities. The fundamental quantities are mass, length, time and temperature, designated by the letters, M, L, T, θ respectively. Temperature is specially useful in compressible flow. The quantities which are expressed in terms of the fundamental or primary quantities are called derived or secondary quantities, (e.g., velocity, area, acceleration etc.). The expression for a derived quantity in terms of the primary quantities is called the dimension of the physical quantity.

A quantity may either be expressed dimensionally in M-L-T or F-L-T system (some engineers prefer to use force instead of mass as fundamental quantity because the force is easy to measure). Table 7.1 gives the dimensions of various quantities used in both the systems.

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Table 7.1: Quantities used in Fluid and Heat transfer and their Dimensions

Table 7.1. Quantities used in Fluid Mechanics and Heat Transfer and their Dimensions

S.No.	Quantity	Dimensions	
		M-L-T System	F-L-T System
	(a) **Fundamental Quantities**		
1.	Mass, M	M	$FL^{-1}T^{2}$
2.	Length, L	L	L
3.	Time, T	T	T
	(b) **Geometric Quantities**		
4.	Area, A	L^{2}	L^{2}
5.	Volume, ~~V~~	L^{3}	L^{3}
6.	Moment of inertia	L^{4}	L^{4}
	(c) **Kinematic Quantities**		
7.	Linear velocity, u, V, U	LT^{-1}	LT^{-1}
8.	Angular velocity, ω; rotational speed, N	T^{-1}	T^{-1}
9.	Acceleration, a	LT^{-2}	LT^{-2}
10.	Angular acceleration, α	T^{-2}	T^{-2}
11.	Discharge, Q	$L^{3}T^{-1}$	$L^{3}T^{-1}$
12.	Gravity, g	LT^{-2}	LT^{-2}
13.	Kinematic viscosity, ν	$L^{2}T^{-1}$	$L^{2}T^{-1}$
14.	Stream function, ψ, circulation, Γ	$L^{2}T^{-1}$	$L^{2}T^{-1}$
15.	Vorticity, Ω	T^{-1}	T^{-1}
	(d) **Dynamic Quantities**		
16.	Force, F	MLT^{-2}	F
17.	Density, ρ	ML^{-3}	$FL^{-4}T^{2}$
18.	Specific weight, w	$ML^{-2}T^{-2}$	FL^{-3}
19.	Dynamic viscosity, μ	$ML^{-1}T^{-1}$	$FL^{-2}T$
20.	Pressure, p; shear stress, τ	$ML^{-1}T^{-2}$	FL^{-2}
21.	Modulus of elasticity, E, K	$ML^{-1}T^{-2}$	FL^{-2}
22.	Momentum	MLT^{-1}	FT
23.	Angular momentum or moment of momentum	$ML^{2}T^{-1}$	FLT
24.	Work, W; energy, E	$ML^{2}T^{-2}$	FL
25.	Torque, T	$ML^{2}T^{-2}$	FL
26.	Power, P	$ML^{2}T^{-3}$	FLT^{-1}
	(e) **Thermodynamic Quantities**		
27.	Temperature	θ	θ
28.	Thermal conductivity	$MLT^{-3}\theta^{-1}$	$FT^{-1}\theta^{-1}$
29.	Enthalpy per unit mass	$L^{2}T^{-2}$	$L^{2}T^{-2}$

Example. 7.1. Determine the dimensions of the following quantities:
(i) Discharge, ***(ii)*** Kinematic viscosity, ***(iii)*** Force, and ***(iv)*** Specific weight.

Solution. *(i)* *Discharge* = Area × velocity

$$= L^2 \times \frac{L}{T} = \frac{L^3}{T} = \mathbf{L^3T^{-1}} \text{ (Ans.)}$$

(ii) Kinematic viscosity $(\nu) = \frac{\mu}{\rho}$

where μ is given by: $\tau = \mu \frac{du}{dy}$

$$\therefore \quad \mu = \frac{\tau}{\frac{du}{dy}} = \frac{\text{Shear stress}}{\frac{L}{T} \times \frac{1}{L}} = \frac{\text{Force/area}}{1/T}$$

$$= \frac{\text{Mass} \times \text{acceleration}}{\text{Area} \times 1/T} = \frac{M \times \frac{L}{T^2}}{L^2 \times \frac{1}{T}} = \frac{ML}{L^2T^2 \times \frac{1}{T}}$$

$$= \frac{M}{LT} = ML^{-1}T^{-1} \text{ and } \rho = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{L^3} = ML^{-3}$$

$$\therefore \quad \text{Kinematic viscosity } (\nu) = \frac{\mu}{\rho} = \frac{ML^{-1}T^{-1}}{ML^{-3}} = \mathbf{L^2T^{-1}} \text{ (Ans.)}$$

(iii) Force = mass × acceleration

$$= M \times \frac{\text{length}}{\text{time}^2} = \frac{ML}{T^2} = \mathbf{MLT^{-2}} \text{ (Ans.)}$$

(iv) Specific weight

$$= \frac{\text{Weight}}{\text{Volume}} = \frac{\text{Force}}{\text{Volume}} = \frac{MLT^{-2}}{L^3} = \mathbf{ML^{-2}T^{-2}} \text{ (Ans.)}$$

7.3. DIMENSIONAL HOMOGENEITY

A physical equation is the relationship between two or more physical quantities. Any *correctequation* expressing a physical relationship between quantities *must be dimensionally homogeneous and numerically equivalent. Dimensional homogeneity* states that *every term in an equation when reduced to fundamental dimensions must contain identical powers of each dimension*. Let us consider the equation:

$$p = wh$$

$$\text{Dimensions of L.H.S.} = ML^{-1}T^{-2}$$

$$\text{Dimensions of R.H.S.} = ML^{-2}T^{-2} \times L = ML^{-1}T^{-2}$$

$$\text{Dimensions of L.H.S.} = \text{Dimensions of R.H.S.}$$

$\therefore$ Equation $p = wh$ is dimensionally homogeneous; so it can be used in any system of units.

Applications of Dimensional Homogeneity:

The *principle of homogeneity* proves useful in the following ways:

1. It facilitates to determine the dimensions of a physical quantity.
2. It helps to check whether an equation of any physical phenomenon is dimensionally homogeneous or not.
3. It facilitates conversion of units from one system to another.
4. It provides a step towards dimensional analysis which is fruitfully employed to plan experiments and to present the results meaningfully.

Example 7.2. Determine the dimensions of E in the dimensionally homogeneous Einstein's Equation.

$$E = mc^2\left[\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} - 1\right]$$

where v is the velocity and m is the mass.

Solution. Since the expression is dimensionally homogeneous, the term

$$\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \text{ should be dimensionless}$$

i.e., $$[c] = [v] = \frac{L}{T}$$

$$\therefore \quad [E] = m[c]^2 = M\left[\frac{L^2}{T^2}\right] = ML^2T^{-2}$$

i.e. E has the dimensions of **energy. (Ans.)**

7.4. METHODS OF DIMENSIONAL ANALYSIS

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1. Rayleigh's method
2. Buckingham's π-method
3. Bridgman's method
4. Matrix-tensor method
5. By visual inspection of the variables involved
6. Rearrangement of differential equations.

7.4.1. Rayleigh's Method

This method gives a special form of relationship among the dimensionless groups, and has the inherent drawback that it does not provide any information regarding the number of dimensionless groups to be obtained as a result of dimensional analysis. It is used for determining the expression for a variable which depends upon **maximum three or four variables only**. In case the number of independent variables becomes **more than four, then it is very difficult to find the expression for the dependent variable**.

if X is a variable which depends on X_1, X_2, X_3, ...X_n; the functional equation can be written as:
$X = f(X_1, X_2, X_3, \ldots Xn)$
X is a dependent variable, while X_1, X_2, X_3, ...X_n are independent variables. The Eqn. above can also be written as:

$$X = C({X^a}_1, {X^b}_2, {X^c}_3, \ldots {X^n}_n)$$

Example 7.3.

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Find an expression for the drag force on smooth sphere of diameter D, moving with a uniform velocity V in a fluid density ρ and dynamic viscosity μ.

Solution. The force drag F is a function of
(*i*) Diameter D, (*ii*) Velocity V,
(*iii*) Fluid density ρ, and (*iv*) Dynamic viscosity μ.

Mathematically, $F = f(D, V, \rho, \mu)$ or $F = C(D^a \cdot V^b \cdot \rho^c \cdot \mu^d)$ ...(1)

where, C is a non-dimensional constant.

Using M-L-T system the corresponding equation for dimensions is:

$$MLT^{-2} = [CL^a \cdot (LT^{-1})^b \cdot (ML^{-3})^c \cdot (ML^{-1}T^{-1})^d]$$

For dimensional homogeneity the exponents of each dimension on both sides of the equation must be identical. Thus:

For M: $1 = c + d$ ...(*i*)
For L: $1 = a + b - 3c - d$ ...(*ii*)
For T: $-2 = -b - d$ ...(*iii*)

There are *four unknowns* (a, b, c, d) but *equations are three* in number. Therefore, it is not possible to find the values of a, b, c and d. However, *three of them can be expressed in terms of fourth variable which is most important*. Here the role of viscosity is vital one and hence a, b, c are expressed in terms of d (*i.e.* power to viscosity)

$\therefore$ $c = 1 - d$ ... from (*i*)
$b = 2 - d$ ... from (*iii*)

Putting these values in (*i*), we get:

$$a = 1 - b + 3c + d = 1 - 2 + d + 3(1 - d) + d$$
$$= 1 - 2 + d + 3 - 3d + d = 2 - d$$

Substituting these values of exponents in eqn. (1), we get:

$$F = C[D^{2-d} \cdot V^{2-d} \cdot \rho^{1-d} \cdot \mu^d]$$
$$= C[D^2 V^2 \rho(D^{-d} \cdot V^{-d} \cdot \rho^{-d} \cdot \mu^d)] = C\left[\rho D^2 V^2 \left(\frac{\mu}{\rho VD}\right)^d\right]$$
$$= \boldsymbol{\rho D^2 V^2 \phi\left(\frac{\mu}{\rho VD}\right)} \text{ (Ans.)}$$

Example 7.4.

The efficiency η of a fan depends on the density ρ, the dynamic viscosity μ of the fluid, the angular velocity ω, diameter D of the rotor and the discharge Q. Express η in terms of dimensionless parameters.

Solution. The efficiency η of a fan is a function of:

(*i*) Density ρ, (*ii*) Viscosity μ,
(*iii*) Angular velocity ω, (*iv*) Diameter D, and
(*v*) Discharge Q.

Mathematically, $\eta = f(\rho, \mu, \omega, D, Q)$

or, $\eta = C(\rho^a, \mu^b, \omega^c, D^d, Q^e)$ ...(1)

where C is a non-dimensional constant.

Using M-L-T system, the corresponding equation for dimensions is:

$$M^oL^oT^o = C[(ML^{-3})^a (ML^{-1}T^{-1})^b (T^{-1})^c (L)^d (L^3 T^{-1})^e]$$

For dimensional homogeneity the exponents of each dimension on both sides of the equation must be identical. Thus:

For M: $0 = a + b$

For L: $0 = -3a - b + d + 3e$

For T: $0 = -b - c - e$

There are *five variables* and we have only *three equations*. Experience has shown that recognized dimensionless groups appear if the exponents of D, ω and ρ are evaluated in terms of b and e (exponents of viscosity and discharge which are *more important*)

$\therefore$ $a = -b; c = (b + e);$

Continued

$$d = 3a + b - 3e = 3(-b) + b - 3e = -2b - 3e = -(2b + 3e)$$

Substituting these values of exponents in eqn. (1), we get:

$$\eta = C(\rho^{-b} . \mu^b . \omega^{-(b+e)} . D^{-(2b+3e)} . Q^e)$$

$$= C(\rho^{-b} . \mu^b . \omega^{-b} . \omega^{-e} D^{-2b} . D^{-3e} . Q^e)$$

$$= C\left[\left(\frac{\mu}{\rho\omega D^2}\right)^b \left(\frac{Q}{\omega D^3}\right)^e\right]$$

$$= \phi\left[\left(\frac{\mu}{\rho\omega D^2}\right), \left(\frac{Q}{\omega D^3}\right)\right] \text{ (Ans.)}$$

7.4.2. Buckingham's π-Method/Theorem

When a large number of physical variables are involved Rayleigh's method of dimensional analysis becomes *increasingly* laborious and *cumbersome*. Buckingham's method is an *improvement* over Rayleigh's method. Buckingham designated the dimensionless group by the Greek capital letter π(Pi). It is therefore often called *Buckingham **π-method***. The *advantage of this method over Rayleigh's method is that it lets us know, in advance, of the analysis, as to how many dimensionless groups are to be expected.*

Buckingham's π-theorem states as follows:

"If there are n variables (dependent and independent variables) in a dimensionally homogeneous equation and if these variables contain m fundamental dimensions (such as M, L, T, etc.) then the variables are arranged into (n-m) dimensionless terms. These dimensionless terms are called π-terms."

Mathematically, if any variable X_1, depends on independent variables, $X_2, X_3, X_4, \ldots Xn$; the functional equation may be written as: $X_1 = f(X_2, X_3, X_4, \ldots X_n)$ ...(7.3)

Eqn. (7.3) can also be written as: $f_1(X_1, X_2, X_3, \ldots X_n) = 0$ ...(7.4)

It is a dimensionally homogeneous equation and contains n variables. If there are m fundamental dimensions, then according to Buckingham's π-theorem, it [eqn. (7.4)] can be written in terms of number of π-terms (dimensionless groups) in which number of π-terms is equal to (n-m). Hence, eqn. (7.4) becomes as: $f_1(p_1, p_2, p_3 \ldots p_{n-m}) = 0$

Example 7.8:

The resistance R experienced by a partially submerged body depends upon the velocity V, length of the body l, viscosity of the fluid μ, density of the fluid ρ and gravitational acceleration g. Obtain a dimensionless expression for R.

Solution. Step 1. The resistance R is a function of:

(*i*) Velocity V, (*ii*) Length l, (*iii*) Viscosity μ,
(*iv*) Density ρ, and (*v*) Gravitational acceleration g.

Mathematically, $R = f(V, l, \mu, \rho, g)$ ...(*i*)

or, $f_1(R, V, l, \mu, \rho, g) = 0$ ...(*ii*)

∴ Total number of variables, $n = 6$

$$\left\{\begin{array}{l} m \text{ is obtained by writing dimensions of each variable as:} \\ R = MLT^{-2}, V = LT^{-1}, \mu = ML^{-1}T^{-1}, \rho = ML^{-3}, g = LT^{-2}. \text{Thus the} \\ \text{fundamental dimensions in the problem are } M, L, T \text{ and hence } m = 3 \end{array}\right\}$$

Number of dimensionless π-terms = $n - m = 6 - 3 = 3$

Thus three π–terms say π_1, π_2, and π_3 are formed.

The eqn. (*ii*) may be written as:

$$f_1(\pi_1, \pi_2, \pi_3) = 0 \quad \ldots(iii)$$

Step 2. Selection of repeating variables: Out of six variables R, V, l, μ, ρ, g three variables (as $m = 3$) are to be selected as *repeating variables*. R is a dependent variable and should *not* be selected as a repeating variable. Out of the remaining five variables one variable should have *geometric property*, second should have *flow property* and third one should have *fluid property*; these requirements are met by selecting l, V and ρ *as repeating variables*. The repeating variables themselves should not form a dimensionless term and must contain *jointly all fundamental dimensions equal to m i.e.* 3 here. Dimensions of l, V and ρ are L, LT^{-1}, ML^{-3} and hence the three fundamental dimensions exist in l, V and ρ and also *no dimensionless group is formed by them.*

Continued

Step 3. Each π-term (= m + 1 variables) is written as given in eqn. (7.6), *i.e.,*

$$\left.\begin{aligned} \pi_1 &= l^{a_1}.V^{b_1}.\rho^{c_1}.R \\ \pi_2 &= l^{a_2}.V^{b_2}.\rho^{c_3}.\mu \\ \pi_3 &= l^{a_3}.V^{b_3}.\rho^{c_3}.g \end{aligned}\right\} \quad ...(iv)$$

Step 4. Each π-term is solved by the *principle of dimensional homogeneity, as* follows:

π_1-term:

$$\pi_1 = l^{a_1}.V^{b_1}.\rho^{c_1}.R$$
$$M^0L^0T^0 = L^{a_1}.(LT^{-1})^{b_1}.(ML^{-3})^{c_1}.(MLT^{-2})$$

Equating the exponents of M, L and T respectively, we get:

For M: $0 = c_1 + 1$

For L : $0 = a_1 + b_1 - 3c_1 + 1$

For T: $0 = -b_1 - 2$

π_2–term:

$$\pi_2 = 1^{a_2}.V^{b_2}.\rho^{c_2}.\mu$$
$$M^0L^0T^0 = L^{a_2}.(LT^{-1})^{b_2}.(ML^{-3})^{c_2}.(ML^{-1}T^{-1})$$

Equating the exponents of M, L and T respectively, we get:

For M: $0 = c_2 + 1$

For L: $0 = a_2 + b_2 - 3c_2 - 1$

For T: $0 = -b_2 - 1$

$\therefore$ $c_2 = -1$; $b_2 = -1$

and, $a_2 = -b_2 + 3c_2 + 1 = 1 - 3 + 1 = -1$

Substituting the values of a_2, b_2, and c_2 in π_2, we get:

$$\therefore \quad \pi_2 = l^{-1}.V^{-1}.\rho^{-1}.\mu = \frac{\mu}{lV\rho}$$

Continued

π_3-term:

$$\pi_3 = 1^{a_3}.V^{b_3}.\rho^{c_3}.g$$
$$M^0L^0T^0 = L^{a_3}.(LT^{-1})^{b_3}.(ML^{-3})^{c_3}.(LT^{-2})$$

Equating the exponents of M, L and T respectively, we get:

For M: $0 = c_3$

For L: $0 = a_3 + b_3 - 3c_3 + 1$

For T: $0 = -b_3 - 2$

$\therefore$ $c_3 = 0$; $b_3 = -2$

and, $a_3 = -b_3 + 3c_3 - 1 = 2 + 0 - 1 = 1$

Substituting the values of a_3, b_3, and c_3 in π_3, we get:

$$\therefore \quad \pi_3 = l^1.V^{-2}.\rho^0.g = \frac{lg}{V^2}$$

Step 5. Substitute the values of π_1, π_2, π_3 in eqn. (*iii*). The functional relationship becomes:

$$f_1\left(\frac{R}{l^2V^2\rho}, \frac{\mu}{lV\rho}, \frac{lg}{V^2}\right) = 0$$

or,

$$\frac{R}{l^2V^2\rho} = \phi\left(\frac{\mu}{lV\rho}, \frac{lg}{V^2}\right) = \phi\left(\frac{\rho Vl}{\mu}, \frac{V}{\sqrt{lg}}\right)$$

The above step has been made on the postulate that *reciprocal of pi-term and its square root is non-dimensional.*

$$R = l^2V^2\rho\,\phi\left(\frac{\rho Vl}{\mu}, \frac{V}{\sqrt{lg}}\right) \quad \textbf{(Ans.)}$$

Example 7.9.

Using Buckingham's π-theorem, show that the velocity through a circular orifice is given by $V = 17\sqrt{2gH}\varphi\left[\frac{D}{H}, \frac{\mu}{\rho V H}\right]$

where,
H = Head causing flow,
D = Diameter of the orifice,
μ = Co-efficient of viscosity,
ρ = Mass density, and
g = Acceleration due to gravity

Solution. V is a function of: H, D, μ, ρ and g

Mathematically, $V = f(H, D, \mu, \rho, g)$ ...(i)

or, $f_1(V, H, D, \mu, \rho, g) = 0$ ...(ii)

∴ Total number of variables, $n = 6$

Writing dimensions of each variable, we have:

$$V = LT^{-1}, H = L, D = L, \mu = ML^{-1}T^{-1}, \rho = ML^{-3}, g = LT^{-2}$$

Thus, number of fundamental dimensions, $m = 3$

∴ Number of π-terms $= n - m = 6 - 3 = 3$

Eqn. (ii) can be written as:

$$f_1(\pi_1, \pi_2, \pi_3) = 0 \quad ...(iii)$$

Each π-term contains $(m + 1)$ variables, where $m = 3$ and is also equal to repeating variables. Choosing H, g, ρ as *repeating variables* (V being a dependent variable should not be chosen as repeating variable), we get three π-terms as:

$$\pi_1 = H^{a_1} . g^{b_1} \rho^{c_3} . V$$
$$\pi_2 = H^{a_2} . g^{b_2} \rho^{c_2} . D$$
$$\pi_3 = H^{a_3} . g^{b_3} \rho^{c_3} . \mu$$

Continued

π_1-term:

$$\pi_1 = H^{a_1} . g^{b_1} \rho^{c_1} . V$$
$$M^0L^0T^0 = L^{a_1} . (LT^{-2})^{b_1} . (ML^{-3})^{c_1} . (LT^{-1})$$

Equating the exponents of M, L and T respectively, we get:

For M: $0 = c_1$

For L: $0 = a_1 + b_1 - 3c_1 + 1$

For T: $0 = -2b_1 - 1$

∴ $c_1 = 0; b_1 = -\frac{1}{2}$

and, $a_1 = -b_1 + 3c_1 - 1 = \frac{1}{2} + 0 - 1 = -\frac{1}{2}$

Substituting the values of a_1, b_1 and c_1 in π_1, we get:

∴ $$\pi_1 = H^{-1/2} . g^{-\frac{1}{2}} . \rho^0 . V = \frac{V}{\sqrt{gh}}$$

π_2-term:

$$\pi_2 = H^{a_2} . g^{b_2} . \rho^{c_2} . D$$
$$M^0L^0T^0 = L^{a_2} . (LT^{-2})^{b_2} . (ML^{-3})^{c_2} . L$$

Equating the exponents of M, L and T respectively, we get:

For M: $0 = c_2$

For L: $0 = a_2 + b_2 - 3c_2 + 1$

For T: $0 = -2b_2$

∴ $c_2 = 0; b_2 = 0$

and, $a_2 = -b_2 + 3c_2 - 1 = -1$

Substituting the values of a_2, b_2, and c_2 in π_2, we get:

$$\pi_2 = H^{-1} . g^0 . \rho^0 . D = \frac{D}{H}$$

Continued

π_3-term:

$$\pi_3 = H^{a_3} \cdot g^{b_3} \cdot \rho^{c_3} \cdot \mu$$
$$M^0L^0T^0 = L^{a_3} \cdot (LT^{-2})^{b_3} \cdot (ML^{-3})^{c_3} \cdot ML^{-1}T^{-1}$$

Equating the exponents of M, L and T respectively, we get:

For M: $\quad 0 = c_3 + 1$

For L: $\quad 0 = a_3 + b_3 - 3c_3 - 1$

For T: $\quad 0 = -2b_3 - 1$

$\therefore \quad c_3 = -1;\ b_3 = -\frac{1}{2}$

and, $\quad a_3 = -b_3 + 3c_3 + 1 = \frac{1}{2} - 3 + 1 = -\frac{3}{2}$

Substituting the values of a_3, b_3, and c_3 in π_3, we get:

$$\pi_3 = H^{-3/2} \cdot g^{-\frac{1}{2}} \cdot \rho^{-1} \cdot \mu = \frac{\mu}{H^{3/2}\rho\sqrt{g}}$$

$$= \frac{\mu}{H\rho\sqrt{gH}} = \frac{\mu V}{H\rho V\sqrt{gH}} \qquad \text{(Multiply and divide by } V\text{)}$$

$$= \frac{\mu}{H\rho V} \cdot \pi_1 \qquad \left(\because \frac{V}{\sqrt{gH}} = \pi_1\right)$$

Substituting the values of π_1, π_2 and π_3 in eqn. (*iii*), we get:

$$f_1\left(\frac{V}{\sqrt{gH}}, \frac{D}{H}, \frac{\mu}{HlV} \cdot \pi_1\right) = 0$$

or, $$\frac{V}{\sqrt{gH}} = \phi\left[\frac{D}{H}, \frac{\mu}{H\rho V} \cdot \pi_1\right]$$

or, $$V = \sqrt{2gH}\,\phi\left[\frac{D}{H}, \frac{\mu}{\rho VH}\right] \qquad \text{...Proved.}$$

(*Multiplying or dividing by any constant does not change the character of π-terms*).

Example 7.10.

Show that the lift FL on airfoil can be expressed as

$$F_L = \rho V^2 d^2 \phi\left(\frac{\rho V D}{\mu}, \alpha\right)$$

where, ρ = Mass density, V= Velocity of flow, d=a characteristic depth, α = Angle of incidence, and μ = Co-efficient of viscosity.

Solution. Lift F_L is a function of: ρ, V, d, μ, α

Mathematically, $\quad F_L = f(\rho, V, d, \mu, \alpha) \quad$ ...(*i*)

or, $\quad f_1(F_L, \rho, V, d, \mu, \alpha) \quad$ ...(*ii*)

$\therefore$ Total number of variables, $n = 6$

Writing dimensions of each variable, we have:

$$F_L = MLT^{-2},\ \rho = ML^{-3},\ V = LT^{-1},\ d = L,\ \mu = ML^{-1}T^{-1},\ \alpha = M^0L^0T^0$$

Thus, number of fundamental dimensions, $m = 3$

$\therefore \quad$ Number of π-terms $= n - m = 6 - 3 = 3$

Eqn. (*ii*) can be written as: $f_1(\pi_1, \pi_2, \pi_3) = 0 \quad$ ...(*iii*)

Each π-term contains $(m + 1)$ variables, where $m = 3$ and is also equal to repeating variables. Choosing d, V and ρ as repeating variables, we get these π-terms as:

$$\pi_1 = d^{a_1} \cdot V^{b_1} \cdot \rho^{c_1} \cdot F_L$$
$$\pi_2 = d^{a_2} \cdot V^{b_2} \cdot \rho^{c_2} \cdot \alpha$$
$$\pi_3 = d^{a_3} \cdot V^{b_3} \cdot \rho^{c_3} \cdot \mu$$

Continued

π_1-term:

$$\pi_1 = d^{a_1} . V^{b_1} . \rho^{c_1} . F_L$$
$$M^0L^0T^0 = L^{a_1} . (LT^{-1})^{b_1} . (ML^{-3})^{c_1} . (MLT^{-2})$$

Equating the exponents of M, L and T respectively, we get:

For M: $0 = c_1 + 1$

For L: $0 = a_1 + b_1 - 3c_1 + 1$

For T: $0 = -b_1 - 2$

$\therefore$ $c_1 = -1; b_1 = -2$

$\therefore$ $a_1 = -b_1 + 3c_1 - 1 = 2 - 3 - 1 = -2$

Substituting the values of a_1, b_1 and c_1 in π_1, we get:

$$\pi_1 = d^{-2} . V^{-2} . \rho^{-1} . F_L = \frac{F_L}{\rho V^2 d^2}$$

π_2-term:

$$\pi_2 = d^{a_2} . V^{b_2} . \rho^{c_2} . \mu$$
$$M^0L^0T^0 = L^{a_2} . (LT^{-1})^{b_2} . (ML^{-3})^{c_2} . (ML^{-1}T^{-1})$$

Equating the exponents of M, L and T respectively, we get:

For M: $0 = c_2 + 1$

For L: $0 = a_2 + b_2 - 3c_2 - 1$

For T: $0 = -b_2 - 1$

$\therefore$ $c_2 = -1; b_2 = -1$

and, $a_2 = -b_2 + 3c_2 + 1 = 1 - 3 + 1 = -1$

Substituting the values of a_2, b_2 and c_2 in π_2, we get:

$$\pi_2 = d^{-1} . V^{-1} . \rho^{-1} . \mu = \frac{\mu}{\rho V d}$$

or, $$\pi_2 = \frac{\rho V d}{\mu}$$

Continued

π_3-term:

$$\pi_3 = d^{a_3} . V^{b_3} \rho^{c_3} . \alpha$$
$$M^0L^0T^0 = L^{a_3} . (LT^{-1})^{b_3} . (ML^{-3})^{c_3} . (M^0L^0T^0)$$

Equating the exponents of M, L and T respectively, we get:

For M: $0 = c_3 + 0$

For L: $0 = a_3 + b_3 - 3c_3 + 0$

For T: $0 = -b_3 + 0$

$\therefore$ $c_3 = 0; b_3 = 0$

and, $a_3 = -b_3 + 3c_3 = 0$

Substituting the values of a_3, b_3 and c_3 in π_3, we get:

$$\pi_3 = d^0 . V^0 . \rho^0 . \alpha = \alpha$$

Substituting the values of π_1, π_2 and π_3 in eqn. (*iii*), we get:

$$f_1\left(\frac{F_L}{\rho V^2 d^2}, \frac{\rho V d}{\mu}, \alpha\right)$$

$$\frac{F_L}{\rho V^2 d^2} = \phi\left(\frac{\rho V d}{\mu}, \alpha\right)$$

or, $$F_L = \rho V^2 d^2 \phi\left(\frac{\rho V d}{\mu}, \alpha\right)$$ **...Proved.**

Summary

Dimensional Analysis is a powerful technique in fluid mechanics used to simplify complex physical phenomena by analyzing the dimensions of variables involved. The primary method, **the Buckingham π theorem**, identifies dimensionless groups from the variables influencing a system. These groups, such as the Reynolds number, Mach number, and Froude number, characterize fluid behavior and flow regimes.